

Recap: Some properties of matrices and some definitions.  
We discussed  $|A|$ .

Let  $\underline{A}$  be a square symmetric matrix of the order  $n \times n$ . Consider an equation

~~(\*)~~  $\underline{A} \underline{q} = \lambda \underline{q}$ , where  $\lambda$  is known as an eigenvalue of  $\underline{A}$  with the corresponding eigenvector  $\underline{q}$  ( $\underline{q} \neq 0$ ).

- 1) There are  $n$  eigenvalues (some of them can repeat).
- 2) eigenvectors associated to eigenvalues are orthogonal.
- 3) Eigenvectors can be normalized i.e.  $\underline{q}$  can be chosen such that  $\underline{q}^T \underline{q} = 1$ .

The eigenvalues are solutions to the equation

$$|\underline{A} - \lambda \underline{I}| = 0$$

$$\underline{A} = \begin{bmatrix} 5 & 8 \\ 2 & 3 \end{bmatrix} \Rightarrow |\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 5-\lambda & 8 \\ 2 & 3-\lambda \end{vmatrix}$$

$$\textcircled{2} \quad \begin{vmatrix} 5-\lambda & 8 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(3-\lambda) - 16 = 0$$

$$\Rightarrow 15 - 8\lambda + \lambda^2 - 16 = 0 \Rightarrow \lambda^2 - 8\lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{8 \pm \sqrt{64+4}}{2} \quad \textcircled{1}$$

$$A = \begin{bmatrix} 10 & 3 & 2 \\ 3 & 9 & 3 \\ 2 & 3 & 10 \end{bmatrix}$$

$$\textcircled{1} \quad |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 10 - \lambda & 3 & 2 \\ 3 & 9 - \lambda & 3 \\ 2 & 3 & 10 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda_1 = 15, \lambda_2 = 8, \lambda_3 = 6.$$

$$A \underline{v} = \lambda_1 \underline{v} \quad \underline{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$A \underline{v} = 15 \underline{v} \quad \underline{v} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

~~QED~~

$$\Rightarrow \begin{bmatrix} 10 & 3 & 2 \\ 3 & 9 & 3 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 15 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\Rightarrow 10\alpha_1 + 3\alpha_2 + 2\alpha_3 = 15\alpha_1 \quad \textcircled{1}$$

$$3\alpha_1 + 9\alpha_2 + 3\alpha_3 = 15\alpha_2 \quad \textcircled{2}$$

$$2\alpha_1 + 3\alpha_2 + 10\alpha_3 = 15\alpha_3 \quad \textcircled{3}$$

Fix one of them, then the other two can be expressed in terms of the one we have fixed.

\textcircled{2}

$$\underline{Q} = \begin{bmatrix} \underline{q}_1 : \underline{q}_2 : \underline{q}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\Lambda = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Check:  $\underline{A} = \underline{Q} \Lambda \underline{Q}^T$

this is called the spectral decomposition of a real symmetric matrix.

$$\underline{A} = \underline{Q} \Lambda \underline{Q}^T = \begin{bmatrix} \underline{q}_1 : \underline{q}_2 : \dots : \underline{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \underline{q}_1^T \\ \vdots \\ \underline{q}_n^T \end{bmatrix}$$

$$= \sum_{j=1}^n \lambda_j \underline{q}_j \underline{q}_j^T$$

$$1) \operatorname{tr}(\underline{A}) = \sum_{i=1}^n \lambda_i \quad 2) |\underline{A}| = \prod_{i=1}^n \lambda_i$$

A ~~real~~ symmetric matrix  $\underline{A}$  is not invertible iff at least one of its eigenvalues equals 0.

2)  $\operatorname{rank}(\underline{A}) =$  the number of non-zero eigenvalues.

$$3) \underline{Q} \underline{Q}^T = \underline{Q}^T \underline{Q} = I_n \Rightarrow \underline{Q}^T = \underline{Q}^{-1}$$

$$4) \underline{A}^{-1} = \underline{Q} \Lambda^{-1} \underline{Q}^T \text{ because } \underline{A}^{-1} \underline{A} = \underline{Q} \Lambda^{-1} \underbrace{\underline{Q}^T \underline{Q}}_{I_n} \Lambda \underline{Q}^T = \underline{Q} \underline{Q}^T = I_n$$

③

Now, let us connect the spectral decomposition of a symmetric matrix  $\underline{A}$  with its column space.

Thm: If  $\underline{A}$  is a symmetric matrix, then there exists an orthonormal basis  $\underline{\Phi}$  of  $C(\underline{A})$  consisting of eigenvectors corresponding to non-zero eigenvalues.

Also, eigenvectors of  $\underline{A}$  corresponding to zero eigenvalues are the basis for  $N(\underline{A}^T) = N(\underline{A})$ .

When some of the eigenvalues of  $\underline{A}$  are zero, obviously  $\underline{A}^{-1}$  doesn't exist.

$$\underline{A} = \underline{\Phi} \underline{\Lambda} \underline{\Phi}^T \quad \underline{\Lambda} = \begin{pmatrix} \underline{\Lambda}_n & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix}$$

only  $n$  eigenvalues are non-zero and last  $(n-n)$  eigenvalues are zero.  $\underline{\Lambda}_n$  is an  $n \times n$  diagonal matrix.

$$\underline{\Phi} = \begin{bmatrix} \underline{\Phi}_1 : \underline{\Phi}_2 \end{bmatrix}_{\substack{n \times n \\ n \times (n-n)}}$$

$$\begin{aligned} \underline{A} &= \underline{\Phi} \underline{\Lambda} \underline{\Phi}^T = \begin{bmatrix} \underline{\Phi}_1 & \underline{\Phi}_2 \end{bmatrix}_{\substack{n \times n \\ n \times (n-n)}} \begin{bmatrix} \underline{\Lambda}_n & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{\Phi}_1^T \\ \underline{\Phi}_2^T \end{bmatrix} \\ &= \begin{bmatrix} \underline{\Phi}_1 \underline{\Lambda}_n \underline{\Phi}_1^T & \underline{C}_1 \\ \underline{C}_2 & \underline{C}_3 \end{bmatrix} \end{aligned}$$

thus

$$\underline{Q} = \begin{bmatrix} \underline{Q}_1 \\ \vdots \\ \underline{Q}_2 \end{bmatrix}_{n \times n} \quad \underline{\Lambda} = \begin{bmatrix} \Delta_{kk} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{Q} \underline{\Lambda} = \cancel{\underline{Q}}$$

$$\begin{bmatrix} \underline{Q}_1 & \underline{Q}_2 \end{bmatrix} \cancel{\begin{bmatrix} \Delta_{kk} & 0 \\ 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} \underline{Q}_1 & \underline{Q}_2 \end{bmatrix} \begin{bmatrix} \Delta_{kk} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{Q}_1^T \\ \underline{Q}_2^T \end{bmatrix} = \underline{Q}_1 \Delta_{kk} \underline{Q}_1^T$$

$$\text{Thus } \underline{A}^- = \underline{Q}_1 \Delta_{kk}^{-1} \underline{Q}_1^T$$

$$\text{this } \underline{A}^- \text{ satisfies } \underline{A} \underline{A}^- \underline{A} = \underline{A}$$

$$\text{this additionally satisfies } \underline{A}^- \underline{A} \underline{A}^- = \underline{A}^-$$

### Positive definite matrix

A symmetric matrix  $\underline{A}$  is called a positive definite matrix iff for every vector  $\underline{x} \neq 0$

$$\underline{x}^T \underline{A} \underline{x} > 0$$

$\underline{A}$  is called nonnegative definite iff

$$\underline{x}^T \underline{A} \underline{x} \geq 0 \text{ for all } \underline{x}.$$

$\underline{x}^T \underline{A} \underline{x}$  is called a quadratic form involving  $n$  variables  $\underline{x} = (x_1, \dots, x_n)$

$$\underline{x}^T \underline{A} \underline{x} = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$$

A is positive definite iff all its eigenvalues are greater than 0.

A is non-negative definite when all its eigenvalues are greater than or equal to 0.

Result: If a matrix is non-negative definite it can be written as  $\underline{A} = \underline{L} \underline{L}^T$  for some square matrix  $\underline{L}$ .

$$\text{Pf: } \underline{A} = \underline{Q} \underline{\Lambda} \underline{Q}^T \quad \underline{\Lambda} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \lambda_i \geq 0$$

$$\textcircled{1} \text{ define } \underline{\Lambda}^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix}$$

$$\textcircled{2} \text{ let } \underline{L} = \underline{Q} \underline{\Lambda}^{1/2} \underline{Q}^T \text{ with this } \underline{L}, \quad \underline{A} = \underline{L} \underline{L}^T$$

$$\underline{L} \underline{L}^T = \underline{Q} \underline{\Lambda}^{1/2} \underbrace{\underline{Q}^T \underline{Q}}_{\textcircled{3}} \underline{\Lambda}^{1/2} \underline{Q}^T \quad \textcircled{4}$$

$$= \underline{Q} \underline{\Lambda} \underline{Q}^T = \underline{A}$$

If  $\underline{A}$  is positive definite  $\Rightarrow \lambda_i > 0$

$\Rightarrow \underline{L} = \underline{Q} \underline{\Lambda}^{1/2} \underline{Q}^T$  has an inverse

$$\underline{L}^{-1} = \underline{Q} \underline{\Lambda}^{-1/2} \underline{Q}^T, \quad \textcircled{5} \quad \underline{\Lambda}^{-1/2} = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & 0 \\ 0 & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix}$$

This  $\underline{L}$  is popularly referred to as the symmetric square root of the matrix  $\underline{A}$ .

It is denoted by  $\underline{A}^{1/2}$

Form of our linear model was

$$\underline{y} = \underline{x}\beta + \underline{\epsilon},$$

$$E[\underline{\epsilon}] = \underline{0}, \quad \text{Cov}(\underline{\epsilon}) = \sigma^2 I_n$$

$$y_i = x_{i1}\beta_1 + \dots + x_{ip}\beta_p + \epsilon_i \quad i=1, \dots, n.$$

$$E[\epsilon_i] = 0, \quad \text{and} \quad \text{Var}(\epsilon_i) = \sigma^2 \quad \text{and} \quad \text{cov}(\epsilon_i, \epsilon_j) = 0 \\ \text{for } i \neq j$$

$$\Rightarrow E[\underline{y}] = \underline{x}\beta. \quad (\text{Gauss-Markov model assumptions})$$

$$\text{Cov}(\underline{y}) = \sigma^2 I_n.$$

④ Note the facts.

$$\textcircled{1} \quad E[\underline{a}^T \underline{y}] = \underline{a}^T E[\underline{y}] = \underline{a}^T \underline{x}\beta$$

$$\textcircled{2} \quad \text{Var}(\underline{a}^T \underline{y}) = \underline{a}^T \text{Var}(\underline{y}) \underline{a} = \underline{a}^T \sigma^2 I \underline{a} = \sigma^2 \underline{a}^T \underline{a}.$$

$$\textcircled{3} \quad \text{Cov}(\underline{a}^T \underline{y}, \underline{c}^T \underline{y}) = \underline{a}^T \text{Var}(\underline{y}) \underline{c}$$

$$\textcircled{4} \quad \text{Var}(\underline{A}^T \underline{y}) = \underline{A}^T \text{Var}(\underline{y}) \underline{A}$$

Consider the ~~Gauss~~ linear model with the Gauss-Markov model assumptions. Suppose  $\underline{\lambda}^T \beta$  is estimable. Let  $\hat{\beta}$  be any solution to the NEs. What is  $E[\underline{\lambda}^T \hat{\beta}]$  and  $\text{Var}(\underline{\lambda}^T \hat{\beta})$ ?

$$\hat{\beta} = (\underline{x}' \underline{x})^{-1} \underline{x}' \underline{y}, \quad \text{as } \underline{x}' \underline{x} \text{ is a symmetric matrix.}$$

Let's take  $(\underline{x}' \underline{x})^{-1}$  that satisfies both

$$(\underline{x}' \underline{x})^{-1} \underline{x}' \underline{x} (\underline{x}' \underline{x})^{-1} = (\underline{x}' \underline{x})^{-1} \quad \text{and}$$

$$(\underline{x}' \underline{x}) (\underline{x}' \underline{x})^{-1} (\underline{x}' \underline{x}) = (\underline{x}' \underline{x}).$$

$$E[\underline{\lambda}^T \hat{\beta}] = E[\underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{x}' \underline{y}]$$

$$= \underline{\lambda}^T (\cancel{\underline{x}' \underline{x}})^{-1} \underline{x}' \underline{x} \beta$$

$\underline{\lambda}^T \beta$  is estimable  $\Rightarrow \underline{\lambda} \in C(\underline{x}^T) \Rightarrow \underline{\lambda} = \underline{x}' \underline{a}$   
for some  $\underline{a}$

$$\Rightarrow E[\underline{\lambda}^T \hat{\beta}] = \underline{a}^T \underline{x}' (\underline{x}' \underline{x})^{-1} \underline{x}' \underline{x} \beta$$

$$= \underline{a}^T \underline{P}_x \underline{x} \beta = \underline{a}^T \underline{x} \beta = \underline{\lambda}^T \beta$$

$\underline{\lambda}^T \hat{\beta}$  is an unbiased estimator of  $\underline{\lambda}^T \beta$ .

$$\text{Var}(\underline{\lambda}^T \hat{\beta}) = \text{Var}(\underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{x}' \underline{y})$$

$$= \underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{x}' \text{Var}(\underline{y}) \underline{x} (\underline{x}' \underline{x})^{-1} \underline{\lambda}$$

$$= \cancel{\sigma^2} \underbrace{\underline{\lambda}^T (\cancel{\underline{x}' \underline{x}})^{-1} \underline{x}' \underline{x} (\underline{x}' \underline{x})^{-1} \underline{\lambda}}_{\underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{\lambda}}$$

$$= \sigma^2 \underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{\lambda}$$

Gauss-Markov theorem

Under the assumption of the Gauss-Markov model, if  $\underline{\lambda}^T \beta$  is estimable, then  $\underline{\lambda}^T \hat{\beta}$  is the best (minimum variance) linear unbiased estimator (BLUE) of  $\underline{\lambda}^T \beta$ , where  $\hat{\beta}$  solves the NEs.

Pf: Let  $\underline{d}^T \underline{y}$  be any other unbiased estimator of  $\underline{\lambda}^T \beta$ .

$$E[\underline{d}^T \underline{y}] = \underline{\lambda}^T \underline{\beta} + \underline{\beta}$$

$$\Rightarrow \underline{d}^T \underline{x} \underline{\beta} = \underline{\lambda}^T \underline{\beta} + \underline{\beta} \Rightarrow \underline{d}^T \underline{x} = \underline{\lambda}^T$$

$$\text{Var}(\underline{d}^T \underline{y}) = \text{Var}(\underline{d}^T \underline{y} - \underline{\lambda}^T \hat{\underline{\beta}}) + \underline{\lambda}^T \hat{\underline{\beta}}$$

$$= \text{Var}(\underline{d}^T \underline{y} - \underline{\lambda}^T \hat{\underline{\beta}}) + \text{Var}(\underline{\lambda}^T \hat{\underline{\beta}}) + 2 \text{Cov}(\underline{d}^T \underline{y} - \underline{\lambda}^T \hat{\underline{\beta}}, \underline{\lambda}^T \hat{\underline{\beta}})$$

Let us check if  $\text{Cov}(\underline{d}^T \underline{y} - \underline{\lambda}^T \hat{\underline{\beta}}, \underline{\lambda}^T \hat{\underline{\beta}}) = 0$

$$\text{Cov}(\underline{d}^T \underline{y} - \underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{x}' \underline{y}, \underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{x}' \underline{y})$$

Note that  $\underline{\lambda} = \underline{x}' \underline{a}$  for some  $\underline{a}$

$$= \text{Cov}(\underline{d}^T \underline{y} - \underline{a}' \underbrace{\underline{x}' (\underline{x}' \underline{x})^{-1} \underline{x}'}_{\underline{\lambda}} \underline{y}, \underline{a}' \underline{x} (\underline{x}' \underline{x})^{-1} \underline{x}' \underline{y})$$

$$= \text{Cov}(\underline{d}^T \underline{y} - \underline{a}' P_x \underline{y}, \underline{a}' P_x \underline{y})$$

$$= \text{Cov}((\underline{d} - P_x \underline{a})^T \underline{y}, (P_x \underline{a})^T \underline{y})$$

$$= (\underline{d} - P_x \underline{a})^T \text{Var}(\underline{y}) P_x \underline{a}$$

$$= \nabla^T (\underline{d}^T - \underline{a}^T P_x) P_x \underline{a} = \nabla^T (\underline{d}^T P_x \underline{a} - \underline{a}^T P_x \underline{a})$$

~~$$= \nabla^T (\underline{d}^T P_x - \underline{a}^T P_x) \underline{a}$$~~

$$= \nabla^T (\underline{d}^T \underline{x} (\underline{x}' \underline{x})^{-1} \underline{x}' - \underline{a}^T \underline{x} (\underline{x}' \underline{x})^{-1} \underline{x}') \underline{a}$$

$$= \nabla^T (\underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{x}' - \underline{\lambda}^T (\underline{x}' \underline{x})^{-1} \underline{x}') \underline{a} = 0$$

~~$$\Rightarrow \text{Var}(\underline{d}^T \underline{y}) = \text{Var}(\underline{d}^T \underline{y} - \underline{\lambda}^T \hat{\underline{\beta}}) + \text{Var}(\underline{\lambda}^T \hat{\underline{\beta}})$$~~

$$\geq \text{Var}(\underline{\lambda}^T \hat{\underline{\beta}})$$